

The Metre Convention builds on a proposal by the Paris Academy of Sciences from 1791, defining one metre as one ten-millionth of the quarter meridian arc through Paris at a time when angle measurements were more precise than those of length. As metre definitions have evolved with technological advancements over the past 230 years, the loop has now closed—again allowing to define lengths in more accurately defined angle units, Tbl. 1.

Table 1 Evolution of the metre since its beginning.

Year	Fundamental base	Comment	1 metre (by definition)
1791	Terrestrial plane angle	Metre of the Archives	10 μgon
1889	Pt-Ir bar at 0 °C	International Metre	National Metres ±10 μm
1960	Vacuum wavelength	Original SI	1 650 763.73 λ _{Kr-86}
1983	Vacuum speed of light	Fundamental constant	c / 299 792 458 s
2019	Caesium frequency	Phase angle	c / (2π Δν _{Cs}) rad

From $1\text{ m} = \frac{400\text{ gon}}{2\pi}\text{ rad}$ and $1\text{ m} = \frac{c}{2\pi\Delta\nu_{Cs}}\text{ rad}$, cancelling rad and 2π ,

$$400\text{ gon} = \frac{c}{\Delta\nu_{Cs}} \Rightarrow 1\text{ gon} = \frac{c}{400\Delta\nu_{Cs}} \approx 8.16 \times 10^{-5}\text{ m}$$

By contrast, the 1791 definition of the Metre of the Archives was

$$1\text{ gon} = \frac{1}{100} \times 10\,000\,000\text{ m} = 100\,000\text{ m}$$

Hence the original and modern metres differ by a factor of $\frac{10^5}{8.16 \times 10^{-5}} \approx 1.23 \times 10^9$

—an alarming shift of nine orders of magnitude over two centuries!

The classic SI angle quantities are plane angle and solid angle. Following Karush and others, the simplest form of these quantities are:

- plane angle, $\angle$, a figure formed by two rays (half-lines) called arms, with a common endpoint called vertex. An angle divides the plane into two portions; one designated as the interior of $\angle$.
- trihedral angle, $\sphericalangle$, formed by three rays, called edges, with a common initial point called vertex, and the planar surfaces between the edges forming its faces. Polyhedral angles divide space into two portions, one designated as the interior.

Common characteristics of such angles, distinguishing them from quantities expressed in the base units (m, s, kg, A, K, mol, cd) include:

- inherent multidimensionality
- semi-infinite rays emanating from a common vertex
- a minimal and sufficient number of rays to span the space
- division of the n -dimensional space into two distinct regions.

While the 1791 proposal exchanged the degree of angle with the decimal based gon, the Metre Convention is silent on angle units and the SI uses the radian—a powerful mathematical tool, e.g. for computing circle perimeters. In dimension n are similar isoperimetric surfaces, or n -spheres, $\mathcal{S}_{n-1}$ (circle in 2D, sphere in 3D) expressed with equation

$$\mathcal{S}_{n-1} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} r^{n-1}, \quad \begin{cases} \text{integers:} & \Gamma(n) = (n-1)! \\ \text{halves:} & \Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}, \Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4} \end{cases}$$

resulting in the familiar numerical values 2π in 2D and 4π in 3D, Tbl. 2.

Table 2 Properties of n -spheres and related SI units (1980—2024)

n	n -sphere	Perimeter type	Numerical value $\mathcal{S}_{n-1}$	Variable r^{n-1}	In SI base	Angle type	SI angle subdiv.
1	point pair	point	2	constant	m ⁰	phase	$2\pi\text{ rad}^\ddagger$
2	circle	line	2π	r	m	plane	$2\pi\text{ rad}$
3	sphere	surface	4π	r^2	m ²	solid	$4\pi\text{ sr}$
4	3-sphere	volume	$2\pi^2$	r^3	m ³	-	-
5	4-sphere	hypervolume	$8\pi^2/3$	r^4	m ⁴	-	-

[‡]approved coherent phase angle unit 2019

Without detailing the other options, the CIPM in 1980 presented the angle units derived from such surfaces as an interpretation to

- offer flexibility in photometry/radiometry concerning the use of the radian and steradian, following the 1979 redefinition of the candela
- address a conceptual tension already recognized in SI groundwork, later reflected in the now-eliminated class of supplementary units.

This alternative is not without caveats, since for radially defined plane angles, $\sphericalangle = s/r$, and solid angles, $\text{sr} = S/r^2$ with arc, s , and surface, S ,

The Simplex Route to a Consistent Angle System in the SI

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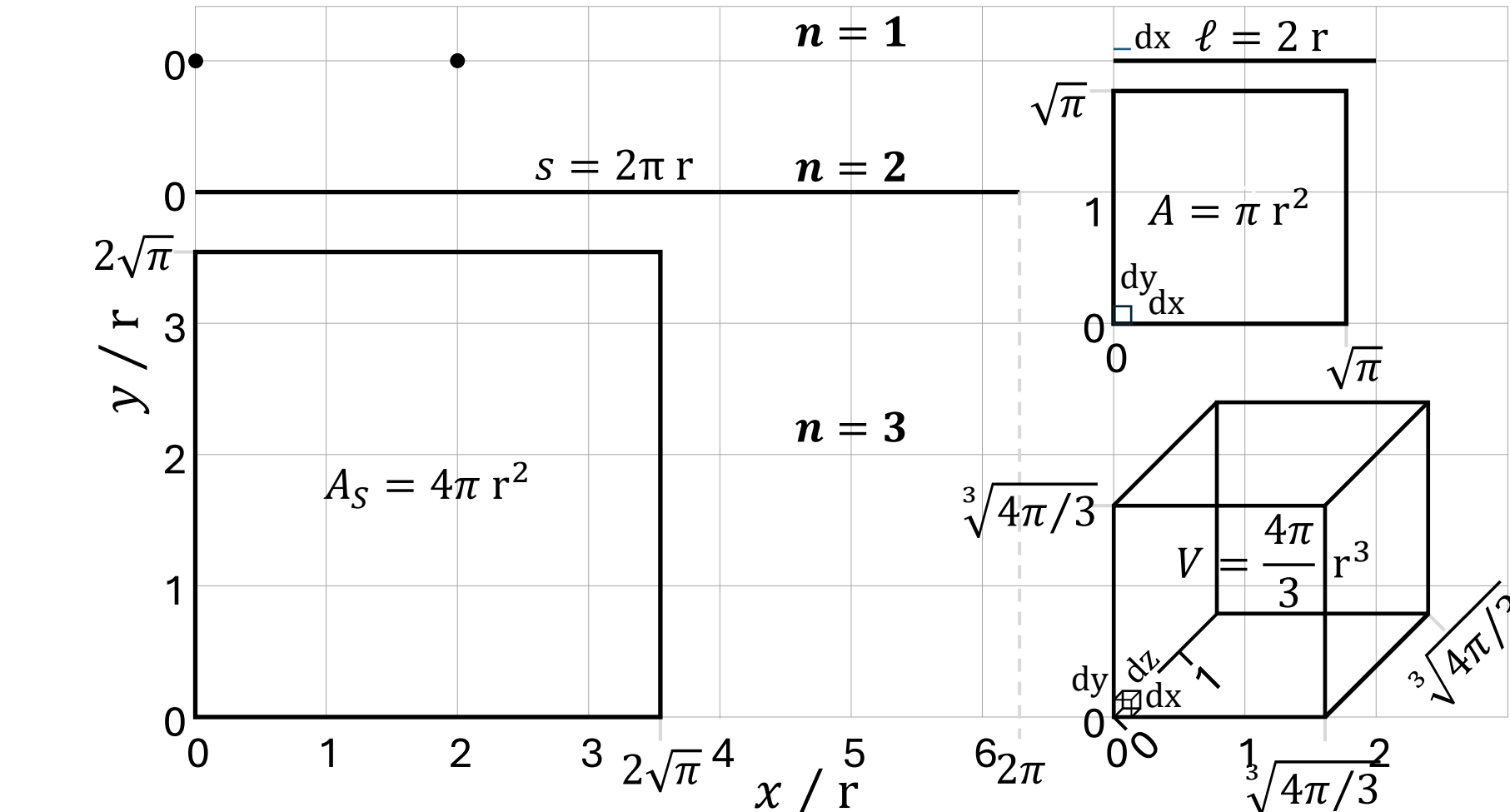


Figure 1 Unitary Cartesian representations of n -sphere boundaries (left) and interiors (right)

- n -boundaries are one dimension short of representing the stipulated portions $dx, dxdy, dxdydz$ of the n -interiors (Fig.1)
- increasing dimensionality scale as $\mathcal{S}_{n-1}$, i.e. $\neq \sqrt{n}$ (Tbl. 2, Fig.1)
- units $\text{rad} = \text{m}/\text{m}$ and $\text{sr} = \text{m}^2/\text{m}^2$ both come out as dimension 1
- the s/r and S/r^2 ratios imply general unity (i.e. $\text{dim}=1$), while e.g., $\lim_{s,r \rightarrow \infty} \sphericalangle(s,r) = \{0 \text{ if } s/r < 1; 1 \text{ if } s/r = 1; \infty \text{ if } s/r > 1\}$ (Fig.2)
- Scaling, e.g., circle sector portions' area A with angle units of dimension 1 is treacherous since $A(r^2, \theta) = \{0.5, 4, 13.5, \dots\} \propto \theta^3$; $A(r, \theta) = \{0.5, 2, 4.5, \dots\} \propto \theta^2$; $A(r \perp \theta) = \{0.5, 1, 1.5, \dots\} \propto \theta$ (Fig.2)
- n -spheres restrict the semi-infinity in the angular quantities' definitions to finite space.

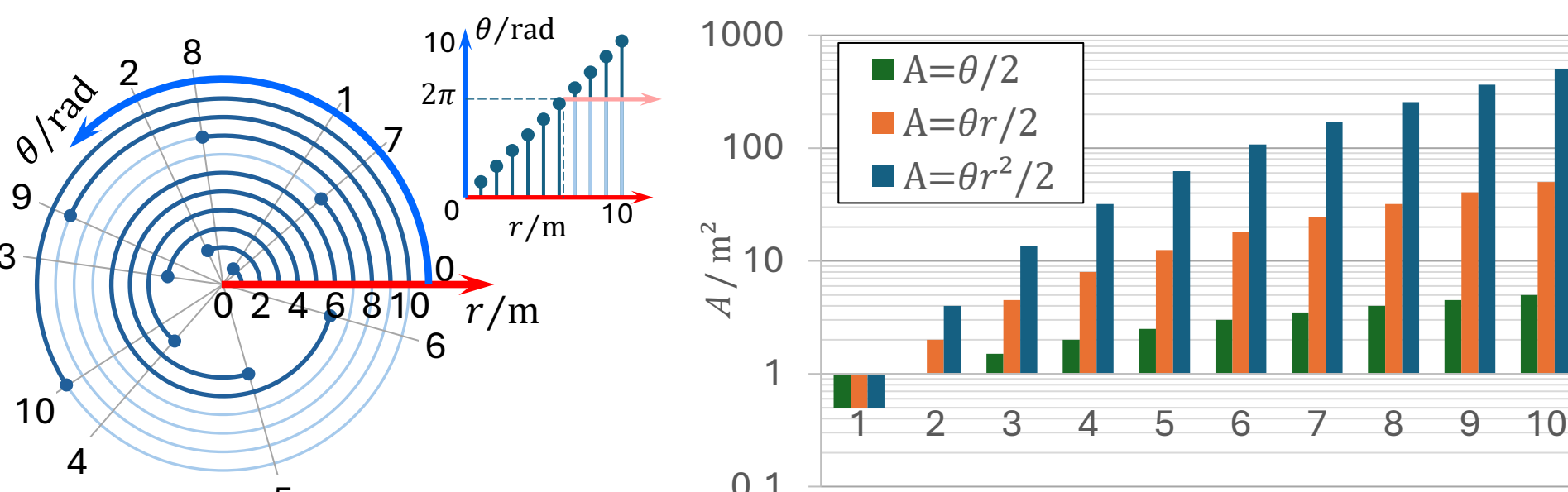


Figure 2 Circle sector areas of a system presented in Polar and Cartesian (r, θ) frames

Like the CIPM that considered plane angle as the ratio of two lengths, Karush conceptually uses circles' central angles in 2D and identifies

- a *radian measure* that relates arc length to radius
- a *numerical measure* that relates angle to a specified fraction of a circle's arc. NB! This is different from the SI, where the characteristics of a degree are inherited from the radian measure.

The numerical approach resolves 4 and 5, but not 1, 2, 3, and 6. Further search identifies an option rooted in right-angled triangles

- a *gradient measure* that relates the length of two legs, i.e $\tan \angle$.
- The gradient resolves 1, 3, 5 and 6, but has similar issues as the radian with 4 and 2, since legs a, b contribute independently and $\lim_{a,b \rightarrow \infty} \angle_{\tan}(a,b) = \{0 \text{ if } b/a < 1; 1 \text{ if } b/a = 1; \infty \text{ if } b/a > 1\}$

Expanding on the gradient measure in Fig.3, we extend the tangent of $p(x, y)$ to the axes in a Cartesian frame and identify

- a *secantial measure* that relates the radial/hypotenuse lengths to its components on the coordinate axes.

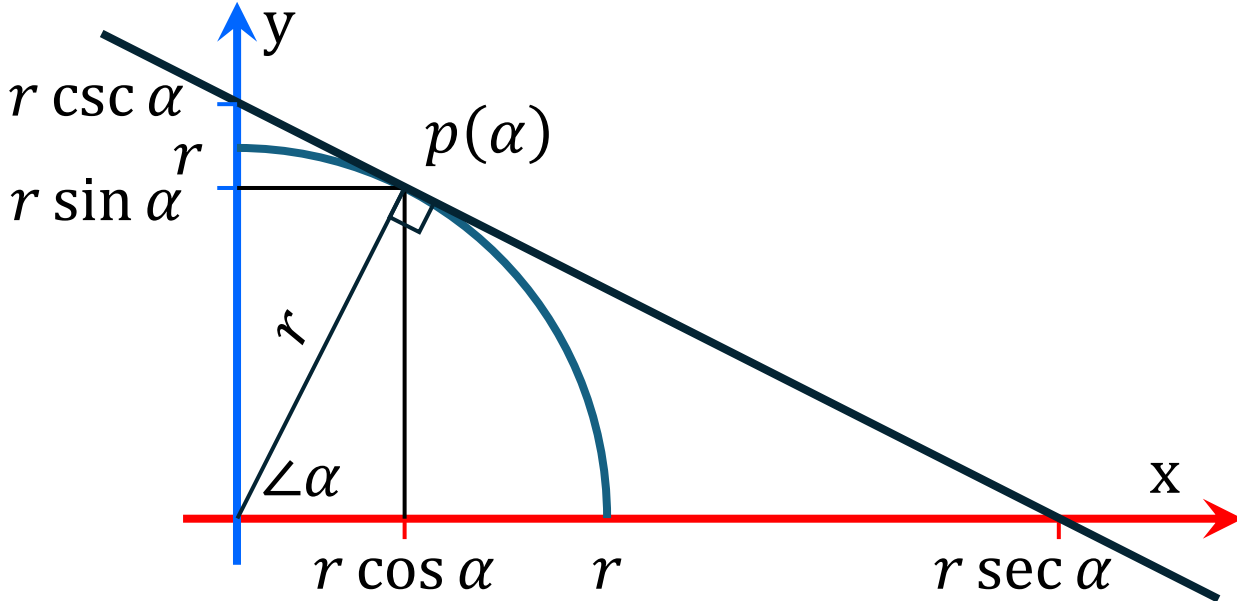


Figure 3 The $\sec \alpha$ and $\csc \alpha$ extend the description of $p(\alpha)$ to $\pm\infty$ in a Cartesian frame

Recalling the properties for zero, right, and straight angles $\perp$ and $\sphericalangle$,

$$\sec \alpha = \frac{1}{\cos \alpha}, \alpha \in (0, \sphericalangle) \text{ with } \sec 0 = 1, \lim_{\alpha \rightarrow \perp} \sec \alpha = \pm\infty, \sec(\sphericalangle) = -1$$
$$\csc \alpha = \frac{1}{\sin \alpha}, \alpha \in (0, \sphericalangle) \text{ with } \lim_{\alpha \rightarrow 0} \csc \alpha = \pm\infty, \csc(\perp) = 1, \lim_{\alpha \rightarrow \sphericalangle} \csc(\alpha) = \pm\infty$$

Table 3. n -simplex features (vertex V, edge E, face F, solid S, pentatope P) and their associated angle quantities of order n .

n	V	E	F	S	P	Geometric figure	Characteristic quantity
0	1	-	-	-	-	point	constant
1	2	1	-	-	-	line segment	phase angle
2	3	3	1	-	-	triangle	plane angle
3	4	6	4	1	-	tetrahedron	solid angle
4	5	10	10	5	1	pentachoron	quadrihedral angle

these ratios are found to resolve all 6 issues related to the isoperimetric units. To confirm the utility of this approach, we note that triangles are the geometric figures with the fewest vertices required to span a plane, with corresponding equivalents— n -simplices—such as the tetrahedron in 3D and the line segment in 1D. Tbl.3 depicts the components of each n -simplex (triangle: 2 dimensions, 3 vertices, 3 edges, 1 face, no volume, etc.) and the corresponding angle characteristics. Relating to Tbl.2, the ones on the diagonal explain how the n -sphere approach takes one entity and divides it by its constituting elements – a triangle equivalent divides its circumference by the summed length of the three sides. Further, by constructing an arbitrary tetrahedron $\triangle ABCD$ in 3D, we examine how this maps to lower dimensions (Fig.4).

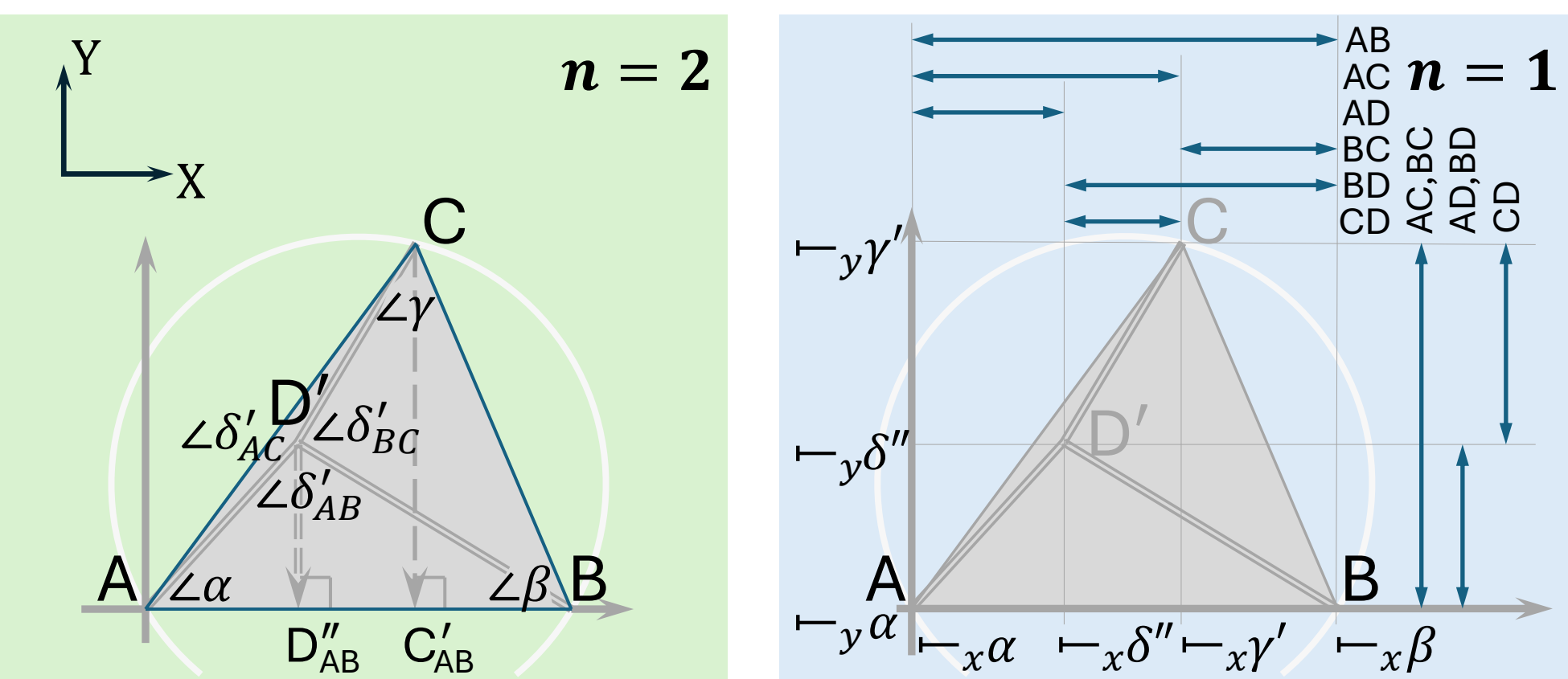
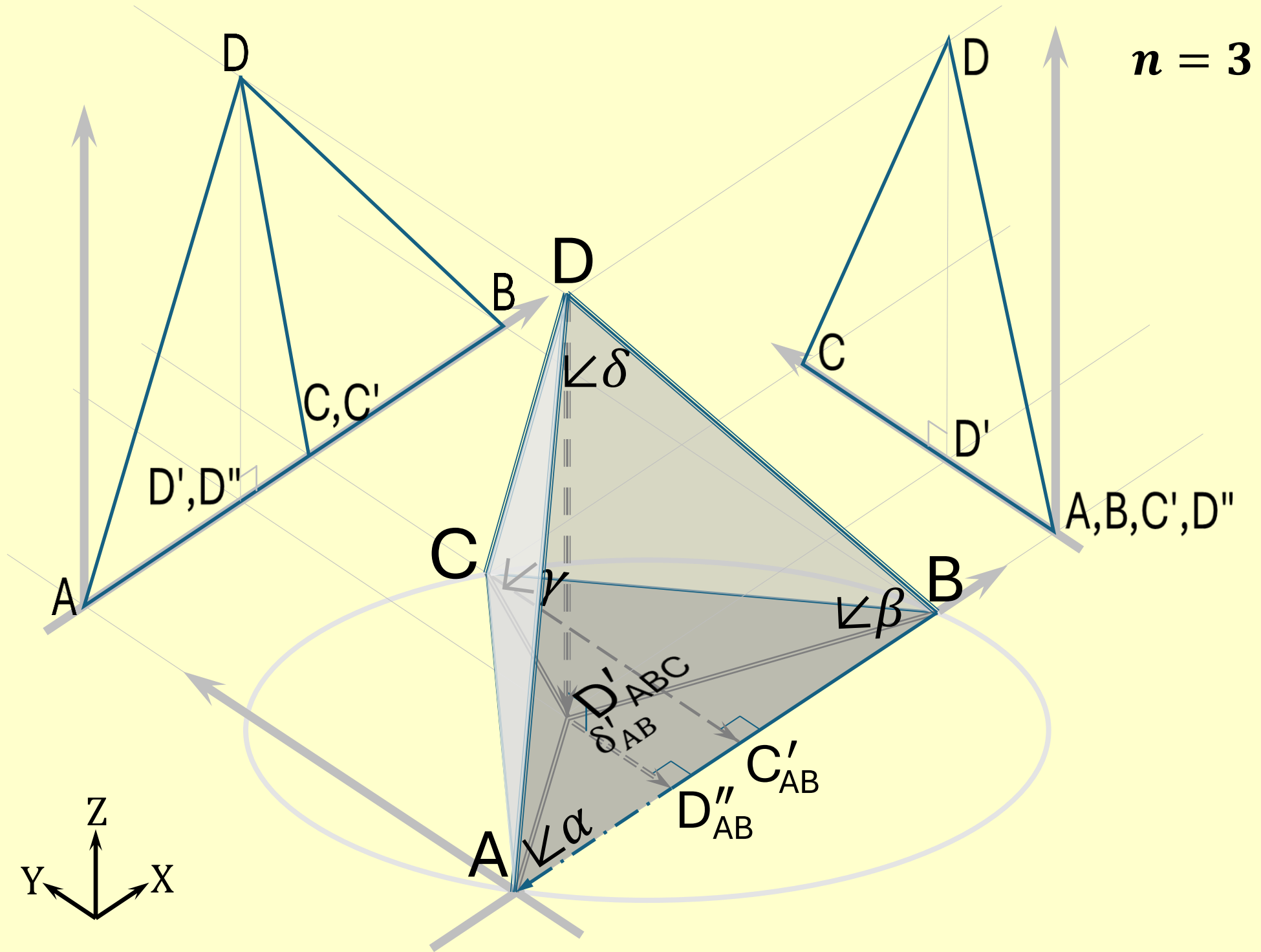


Figure 4 An arbitrary tetrahedron ABCD and its n -dimension angles projected on the plane from the face ABC, the lines AB and $\perp AB$.

From the four available faces of the tetrahedron, we choose $\triangle ABC$ as the reference plane and the line segment $\leftrightarrow AB$ as the primary axis. On this plane and line, D projects as D' and D'' , and C as C' , respectively. The trihedral angles $\angle \alpha, \angle \beta, \angle \gamma$ have three counterparts, $\angle \alpha, \angle \beta, \angle \gamma$ on each plane ABD, ACD, BCD of the sides forming $\triangle ABC$, whereas $\angle \delta$ maps on the reference plane as three different plane angles $\angle \delta'_{AB}, \angle \delta'_{AC}, \angle \delta'_{BC}$ from the projected D' . The analogue mapping of the plane angles on the x- and y-axes brings forth the one-dimensional phase angles $\lrcorner_x \alpha, \lrcorner_y \alpha$, etc.

Using the secantial functions, the tetrahedron's edges are understood in terms of their cardinal components: AB is completely aligned with the x-axis and has no y-component; agnostic of the z-component, AD' is unambiguously shorter than AC and more aligned with the x-axis; BD' is negative and $D'B$ positive in x-direction, etc. Recalling a 1 m bar positioned at one Earth-radius distance, a 0.5 m offset to each end yields the hypotenuse and

$$h = \sqrt{r^2 + 0.5^2} \Rightarrow h/r = \sqrt{1 + \left(\frac{0.5}{r}\right)^2} = \sqrt{1 + \left(\frac{\pi}{40\,000\,000}\right)^2} \approx 1.000\,000\,000\,000\,003$$

a dimensionless scale factor for all measures on the hypotenuse taken perpendicular to the radius—without engaging the arc. Noting that Tbl.3's binomial expansions form a lower-triangular matrix with ones on the diagonal, we recognize that secantial measures abstract angles as the geometric embodiment of correlation in their defining space, just like length serves as a general analogue representation of any 1D quantity. Angles thus emerge as invariant, fundamental relational quantities in n -space—an intuitive interpretation that only requires a unit-norm, orthonormal inner-product framework to be consistently defined in both the Euclidean-perpendicular and quantum- mechanical orthogonality senses.

Table 4 Proposal for a coherent expandable Cartesian Angle System

D	n -space	n -ball	Quantity, symbol	Unit, symbol	Div
1	line	segment	phase angle	$\lrcorner$ phase degree $^\phi$	360
2	plane	disk	plane angle	$\angle, \sphericalangle$ plane degree $^\circ$	360
3	volume	ball	solid angle	$\sphericalangle, \text{sr}$ solid degree $^\bullet$	360
4	4D	4-ball	?	# - ? -	^④ 360

Since the radian was declared the coherent phase angle unit in 2019 and most SI base units are now defined using the frequency $\Delta\nu_{Cs}$ —not directly in time—the SI inadvertently and unnecessarily rests on π , a transcendental constant with infinitely many digits. A Cartesian Angle System (CAS) summarized in Tbl.4, instead promotes a dimensionally consistent n -degree as the proper unit for angles in n -space, with n -degree = $\frac{1}{360}$ of an open n -ball (infinitely extending body, $0 < r \leq \infty$). So defined, phase degree $^\phi$, and plane degree $^\circ$, coincide numerically with today's units, whereas solid degree $^\bullet$, is a genuinely new unit with conversion $1\text{ sr} = \frac{90^\bullet}{\pi}$, similarly to $1\text{ rad} = \frac{180^\circ}{\pi}$. By making degrees the definitional angle units across all dimensions, the CAS establishes angles as unambiguously fundamental quantities—free of the confusing practice of using the same ill-scaled quasi-dimensionless length-ratio units for both length and area, and likewise area-ratio units for volumes. The classic SI units radian and steradian can safely maintain their status as derived in 2D and 3D, but with purely numerical scale factors back to the corresponding degrees. To emphasize the fundamental construct, a direct connection to the number divisible by the lowest three prime and six natural numbers in the definition is compelling. Applying the general SI model since 2019:

The plane degree, symbol $^\circ$, is the SI unit of plane angle. It is defined by taking the fixed numerical value of each angle in an equilateral triangle to be 60.

With minimal disruption, this system

- aligns with general principles for defining angles
- offers precision limited only by technical abilities—not definitions
- reflects how angles are used in both metrology and practice
- clarifies the role of phase angles
- keeps radian and steradian as derived units within the SI
- supports combinations of independent quantities in a unified frame
- reinforces the decimal structure of the SI, while respecting sexagesimal roots
- aspire to be **à tous les temps, à tous les peuples**

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