

NUMERICAL ANALYSIS OF JOSEPHSON JUNCTIONS: A DIDACTIC APPROACH TO SHAPIRO STEPS

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ABSTRACT

This poster provides a didactic overview of the Josephson effect, focusing on the numerical solution of the Resistively and Capacitively Shunted Junction (RCSJ) model and the emergence of Shapiro steps. We present the equivalent electrical circuit, the governing differential equation, and its analogy with a forced pendulum. Through numerical simulations, we solve the differential equation and demonstrate the current-voltage characteristics, highlighting the quantized voltage steps observed under microwave irradiation. The graphical results obtained from these simulations are central to understanding the behavior of Josephson junctions under varying conditions. These findings are crucial for applications in quantum metrology, particularly in the development and adoption of precise voltage standards based on the Josephson effect. This work seeks to bridge theoretical concepts with practical applications, providing an accessible approach for both educational and research purposes. The findings highlight the significance of Josephson junctions in achieving highly precise voltage standards, a cornerstone of modern quantum metrology and a compelling testimony to the ongoing evolution of measurement science.

JOSEPHSON EFFECT

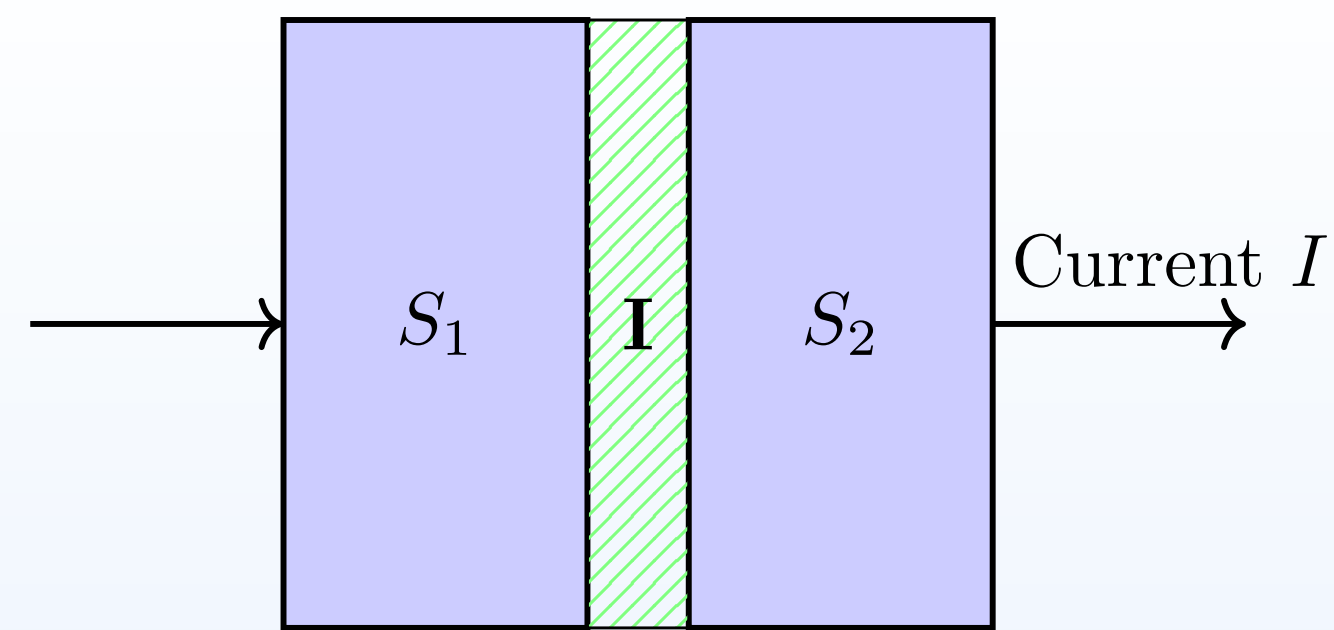


Figure 1: Josephson junction is composed of two superconductors S_1 and S_2 separated by an insulator I .

The Josephson effect governs the flow of supercurrent between two superconductors coupled through a thin insulating barrier. This current arises from Cooper pairs (electron pairs in a macroscopic quantum state) tunneling across the junction. Each superconductor is described by a macroscopic quantum wavefunction with phase ϕ_i , and the voltage V between them relates to the temporal derivative of their phase difference $\phi = \phi_2 - \phi_1$:

$$\frac{d\phi}{dt} = \frac{2eV}{\hbar},$$

where e is the electron charge and $\hbar$ is the reduced Planck constant.

The RCSJ model (Fig. 2) describes the dynamics of a Josephson junction considering its capacitance (C), resistance (R) and Josephson current ($I_J = I_c \sin \phi$), where I_c is the critical current. The total current is $I = I_D + I_R + I_J$. Since $\frac{d\phi}{dt} = \frac{2eV}{\hbar}$, the current equation is:

$$I = \frac{\hbar C}{2e} \frac{d^2\phi}{dt^2} + \frac{\hbar}{2eR} \frac{d\phi}{dt} + I_c \sin \phi$$

The dynamics of a Josephson junction under direct current exhibits a remarkable mechanical analogy to a forced physical pendulum. The differential equation of the forced pendulum,

$$\Gamma = J \frac{d^2\varphi}{dt^2} + \eta \frac{d\varphi}{dt} + \Gamma_c \sin \varphi$$

is directly mapped to the RCSJ equation of the Josephson junction by the following equivalences: torque $\Gamma \leftrightarrow I$, angle $\varphi \leftrightarrow \phi$ (phase difference), moment of inertia $J \leftrightarrow \hbar C/(2e)$, damping $\eta \leftrightarrow \hbar/(2eR)$, and critical torque $\Gamma_c \leftrightarrow I_c$.

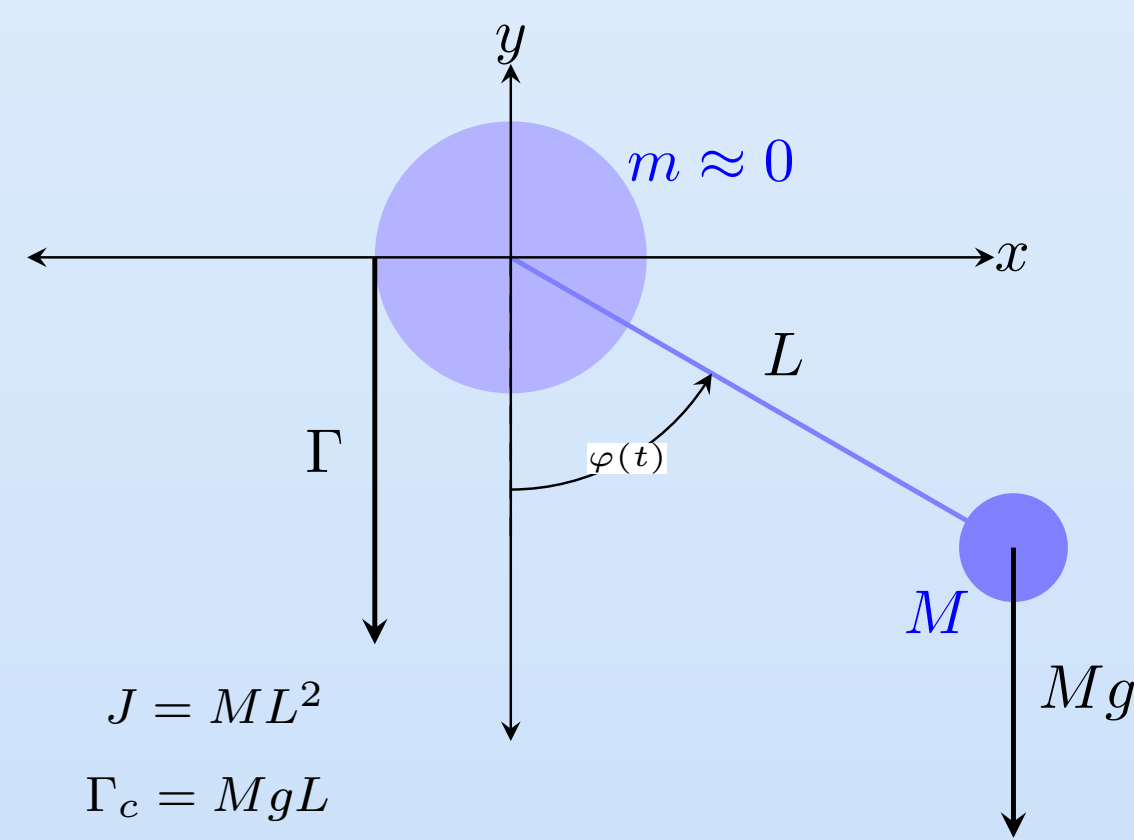


Figure 3: Forced physical pendulum.

PHYSICAL MODEL - SHAPIRO STEPS

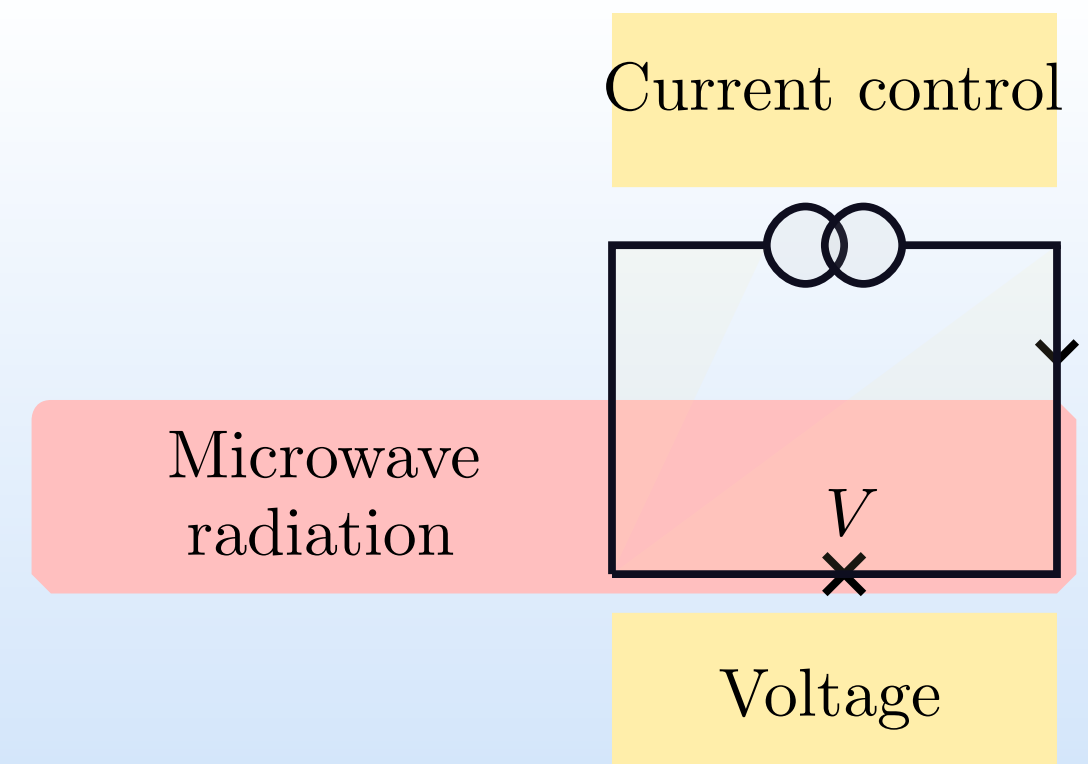


Figure 4: Josephson junction under EM radiation.

If a small alternating voltage component $v \cos(\omega_0 t + \varphi)$ is added to the internal voltage component, equivalent to subjecting the junction to radiation of electromagnetic waves in the microwave range, then the equation of the phase evolution between the two superconductors will have the following form: $\frac{d\phi}{dt} = \frac{2eV}{\hbar} + \frac{2e}{\hbar} v \cos(\omega_0 t + \varphi)$. Considering that $\frac{1}{R} \gg C$, then

$$\frac{\hbar C}{2e} \frac{d^2\phi}{dt^2} + \frac{\hbar}{2eR} \frac{d\phi}{dt} + I_c \sin \phi = I + \frac{v}{R} \cos(\omega_0 t + \varphi)$$

RESULTS OF JOSEPHSON EFFECT

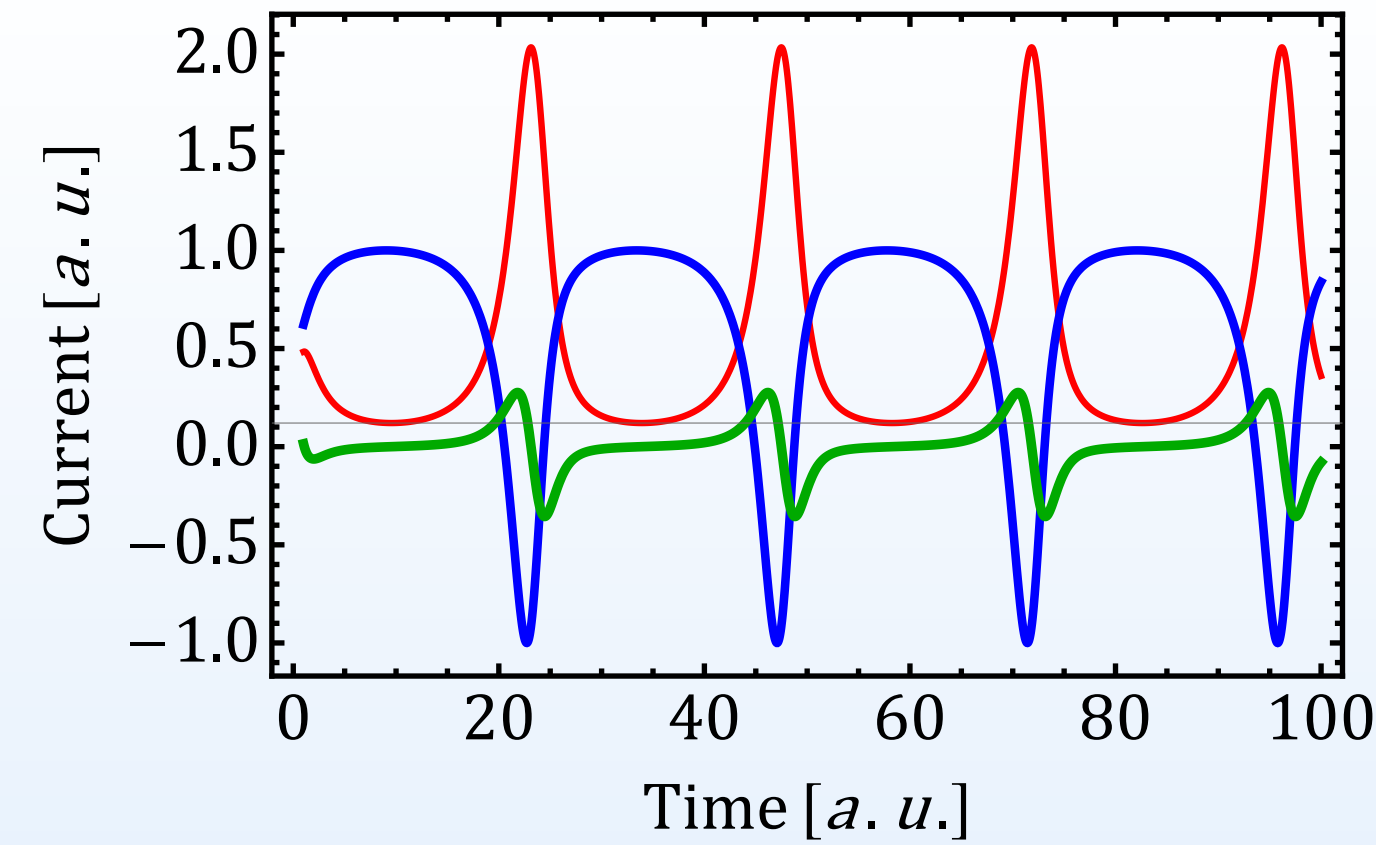


Figure 5: Josephson current (blue), resistive current (red) and displacement current (green) ($I > I_c$).

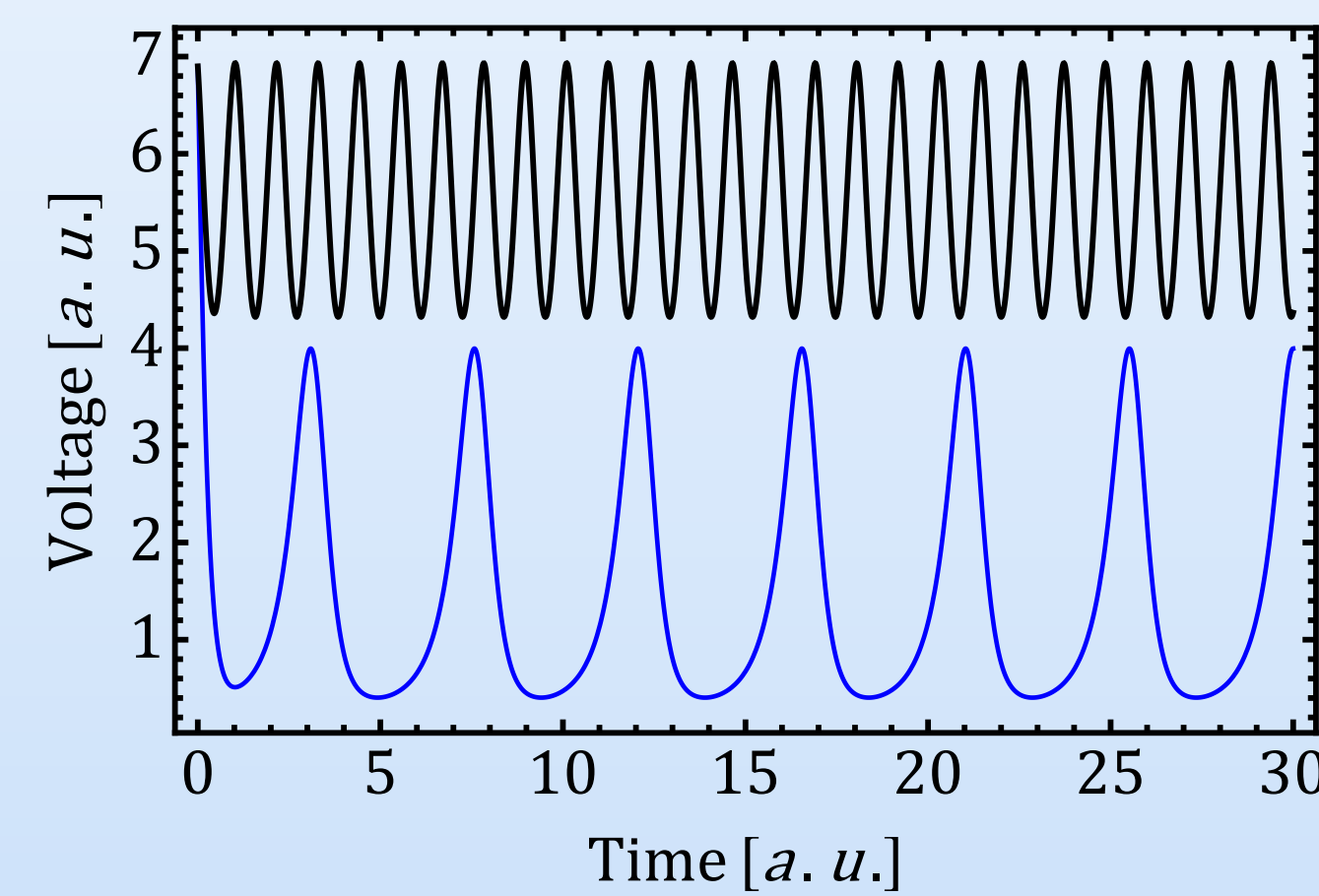


Figure 7: Voltage when $I = 1.2I_c$ (Blue) and $I = 2.9I_c$ (Black).

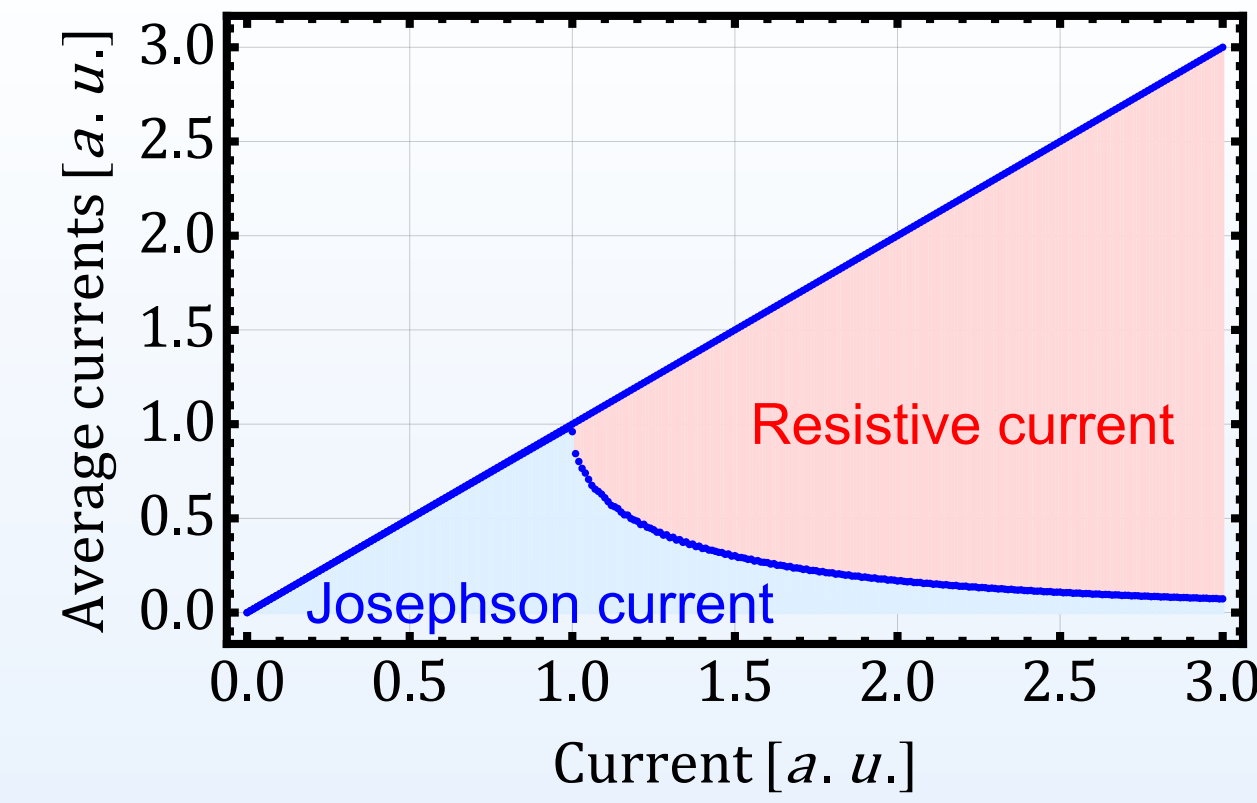


Figure 6: Average Josephson plus resistive current as function of I/I_c ($I_D \approx 0$).

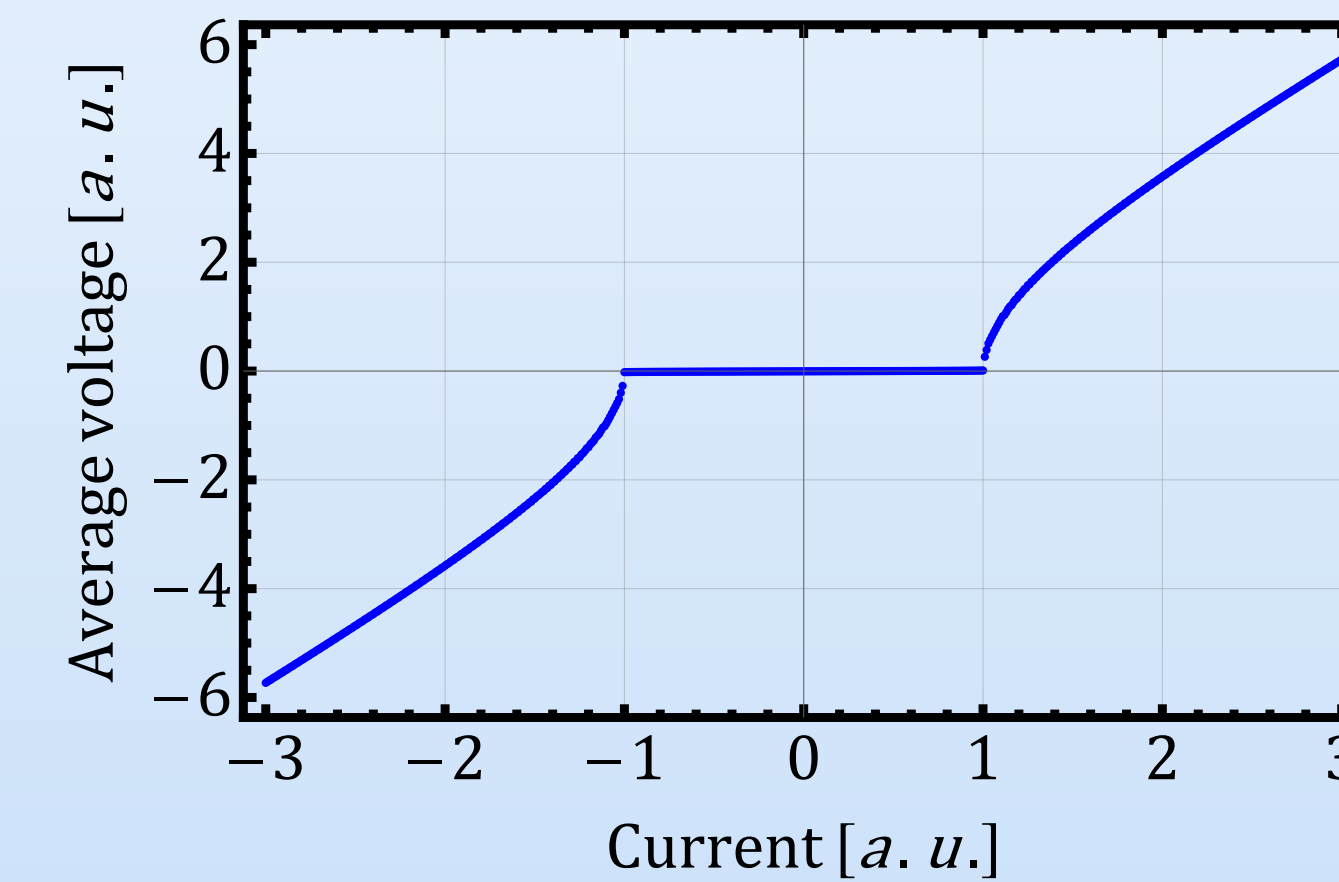


Figure 8: Average voltage as function of I/I_c .

OBTAINING THE SHAPIRO STEPS

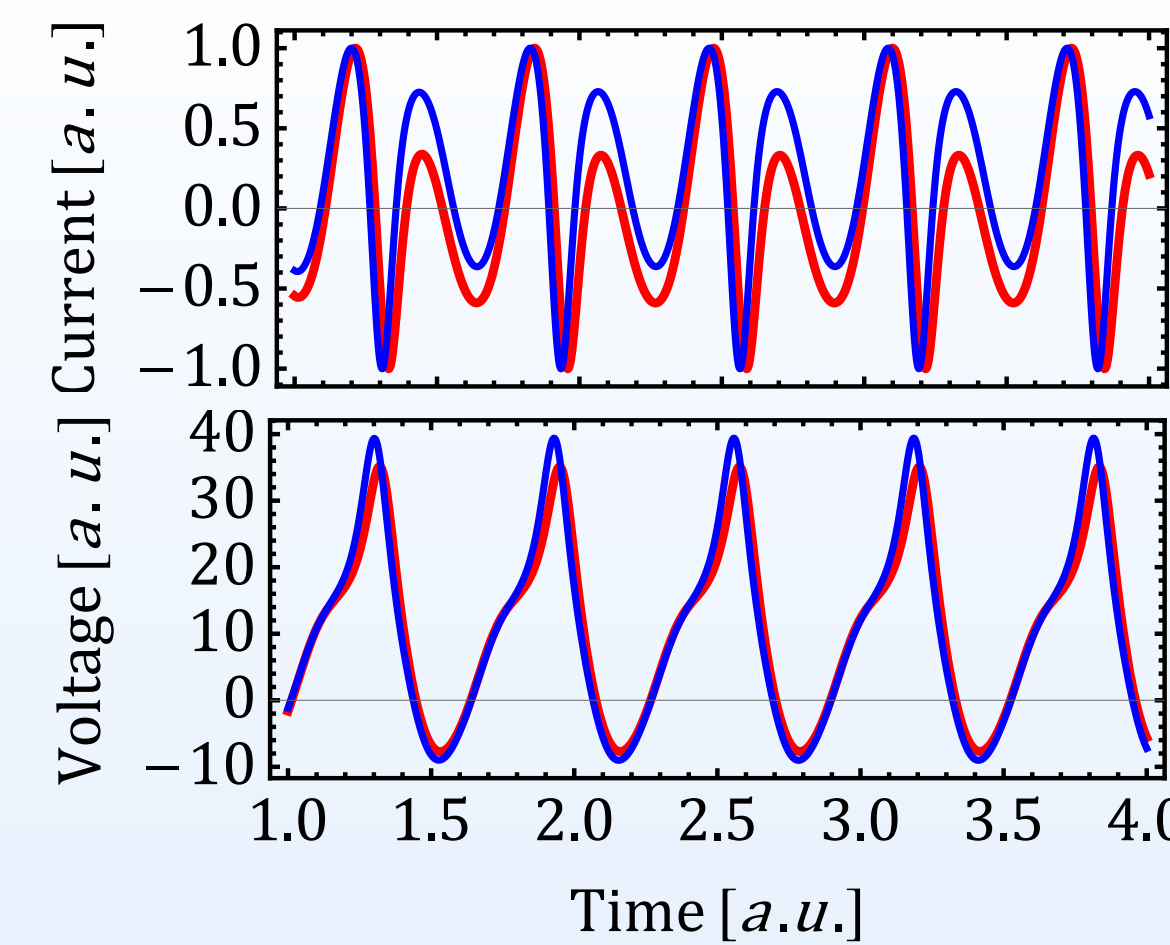


Figure 9: Up: Josephson current when $I_1 = 1.0I_c$ and $I_2 = 1.2I_c$. Down: Voltage $V(I_1) = V(I_2)$.

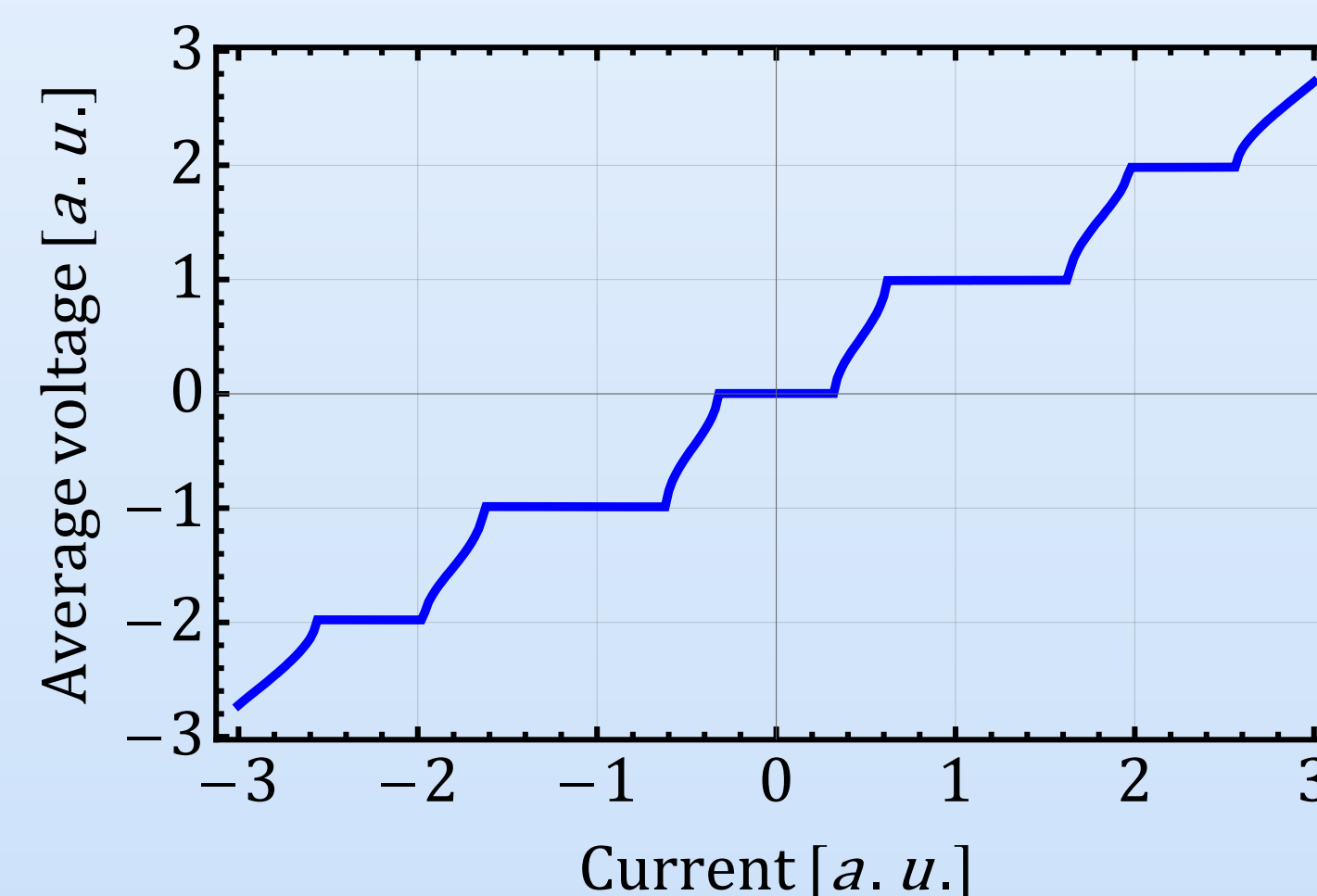


Figure 11: Shapiro steps.

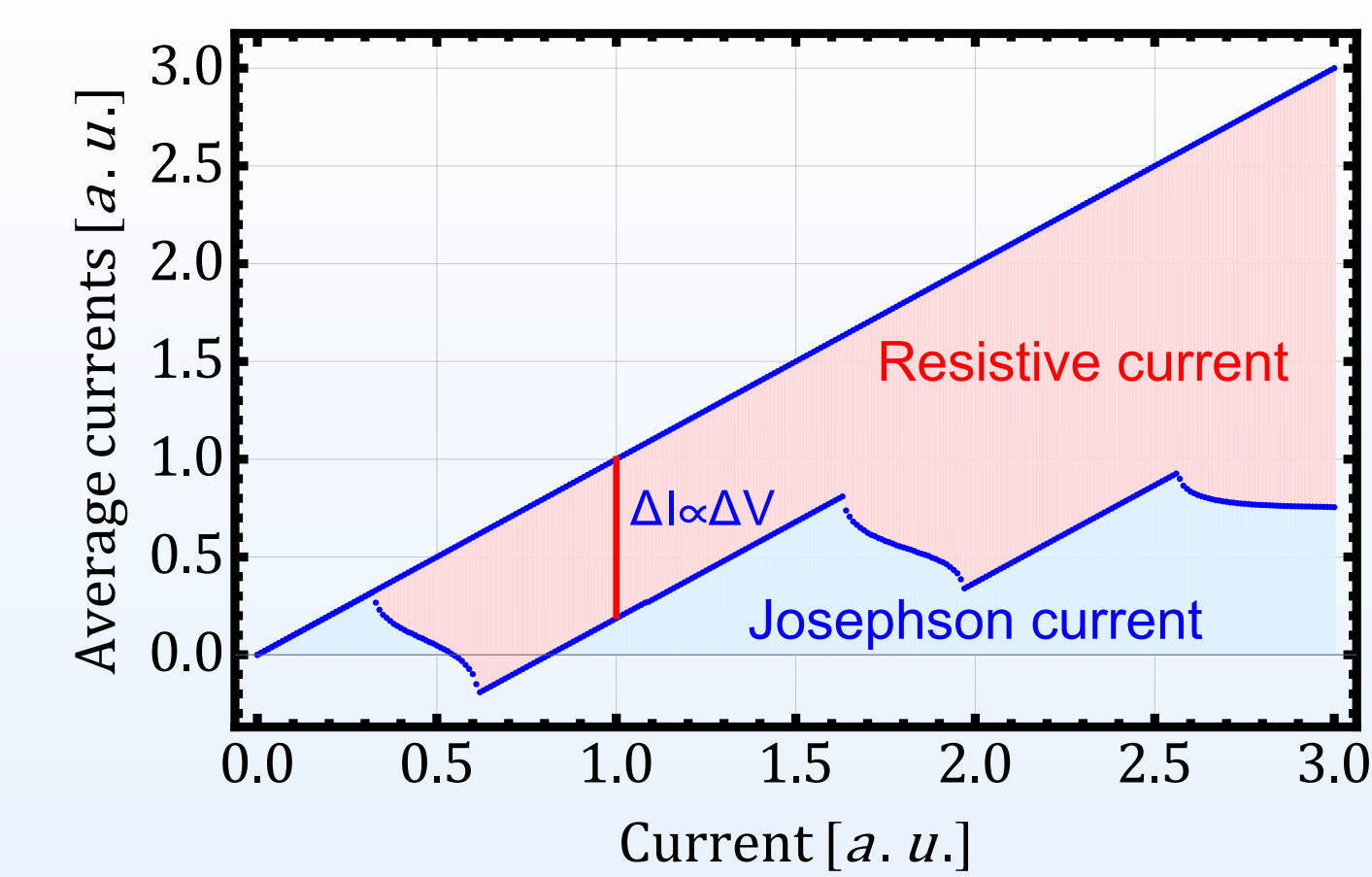


Figure 10: Average Josephson plus resistive current as function of I/I_c ($I_D \approx 0$).

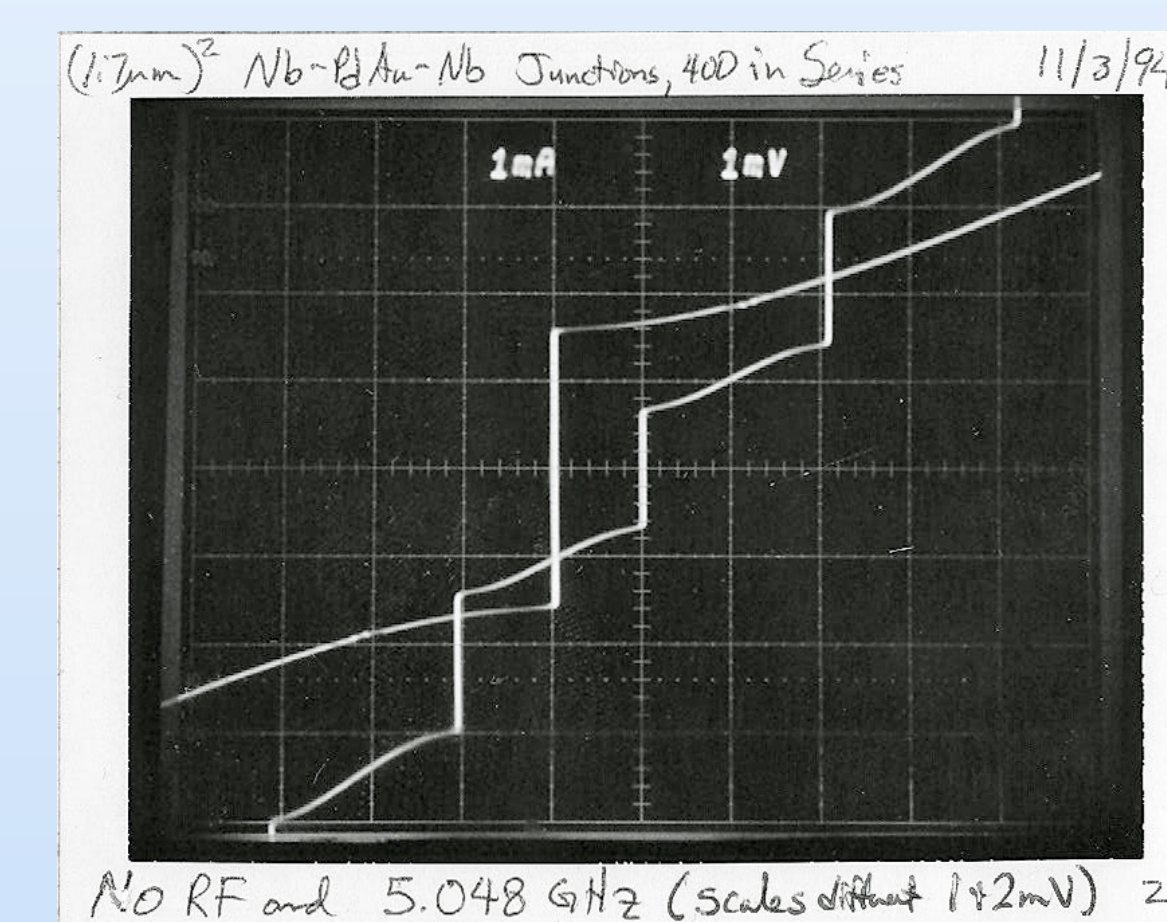


Figure 12: Original photo of oscilloscope traces (double exposure). Source: Benz/NIST.

ANALYSIS AND CONCLUSIONS

Figure 5 demonstrates the conserved current relation-ship: $I_J + I_R + I_D = I$ valid at all times. The Josephson junction displays two distinct operational regimes:

- DC Josephson effect** ($I < I_c$): The entire current flows through the Josephson channel with $I_J = I_c \sin \phi$ and $\langle V \rangle = 0$.
- AC Josephson effect** ($I > I_c$): Voltage oscillations emerge with $V_{\max/\min} = \frac{I \pm I_c}{G}$ and $\langle V \rangle = \frac{\sqrt{I^2 - I_c^2}}{G}$, where $G \equiv 1/R$ denotes the junction conductance.

Figure 11 characterizes the current-voltage response under microwave irradiation (f_0). Increasing current induces discrete **Shapiro steps** in the average voltage with quantized spacing:

$$\Delta V = \frac{h f_0}{2e}$$

arising from phase synchronization between the Josephson frequency $2eV/\hbar$ and external drive harmonics $n\omega_0$. This frequency-locking mechanism produces macroscopic quantum voltage states.

Numerical solutions of the resistively-capacitively shunted junction (RCSJ) model reproduce these Shapiro steps with remarkable fidelity. Such voltage quantization underpins modern metrology, enabling:

- Precision voltage standards via Josephson junction arrays (NIST SNS-type arrays, Fig. 12).
- Digitally programmable outputs through binary array configurations (Fig. 13).

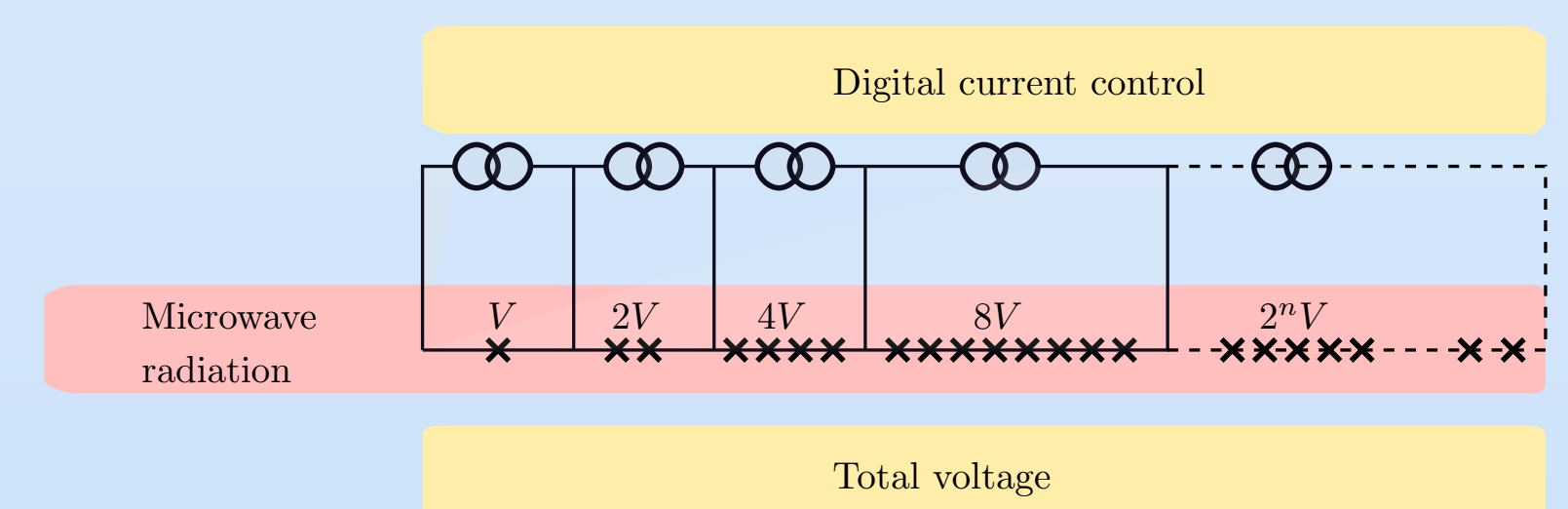


Figure 13: Binary Josephson junction array enabling digital voltage synthesis.

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